Up to now, we've been considering what happens to a block attached to a spring sliding on table

Free body diagram

\[ m \frac{d^2 y}{dt^2} = -b \frac{dy}{dt} - ky + f(t) \]

\[ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0 \]
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\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \]  

**homogeneous eqn**

\[ w = \sqrt{\frac{b}{m}} \quad q = \frac{k}{m} \]

What happens with additional force?

*e.g.* tilting table back and forth

Alternatively, crystal glass tapped with a fork. Position \( y \) represents amount of deformation in the glass. Ringing sound is underdamped.

Freq. of sound is freq. of oscillations due to glass.

External force? Opera singer.

Sound waves push against glass and deform it.

Can opera singer break the glass?

Equation becomes

\[ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + k y = f(t) \]
\[ m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = f(t) \]

or
\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t) \]

with \( p = \frac{b}{m}, q = \frac{k}{m}, g(t) = \frac{f(t)}{m} \)

How to solve?

As for 1st-order linear eqns, we can use the fact that we know how to solve the associated homogeneous eqn

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \]

As before, there's an extended linearity principle

1) If \( y_p(t) \) is a particular solution of the non-homogeneous eqn

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = g(t) \]

Then \( y_h(t) \) is a solution of the homogeneous eqn

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + q y = 0 \]
3. \( y_h(t) \) is a soln of \( \text{assoc'd homog. eqn} \)

then \( y_p + y_h \) is also a soln to the non-homog eqn.

2) If \( y_p \) and \( y_p'(t) \) are two solns to the non-homog. eqn,

then \( y_p - y_p' \) is a soln of the assoc'd homog. eqn.

Therefore, if \( C_1 y_1(t) + C_2 y_2(t) \) is the gen. soln of the homog. eqn, then

\[ y_p(t) + C_1 y_1(t) + C_2 y_2(t) \]

is the general soln to the non-homog. eqn.

In other words, we have a practical
algorithm to solve non-homog. eqns.

Step 1: Find gen soln of assoc’d homog. eqn.

Step 2: Find a particular soln.

Step 3: Combine using extended lin. principle to obtain gen soln of non-homog. eqn.

\[
\begin{align*}
\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y &= e^{-t} \\
\text{Homog. eqn } \quad \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y &= 0 \\
\text{char eqn } \quad r^2 + 5r + 6 &= 0 \\
\Rightarrow (r+2)(r+3) &= 0 \\
100+5s -2 &\neq -3 \\
C_1e^{-2t} + C_2e^{-3t} &\text{ is, gen soln.}
\end{align*}
\]
Step 2

Guess! Does $ae^{-t}$ work?

\[ \frac{d^2(e^{-t})}{dt^2} + 5 \frac{d(e^{-t})}{dt} + 6(e^{-t}) = ae^{-t} - 5ae^{-t} + 6ae^{-t} = ae^{-t} \neq e^{-t} \]

But $ae^{-t}$ works when $a = \frac{1}{2}$

\[ y_p(t) = \frac{1}{2}e^{-t} \text{ is a particular soln} \]

Step 3

\[ \frac{1}{2}e^{-t} + C_1e^{-2t} + C_2e^{-3t} \]

is general soln to non-homog eqn

(Ex.) \[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = e^{-2t} \]

all solns as $t$ increases
Step 1 \[ C_1 e^{-2t} + C_2 e^{-3t} \]

Step 2

Guess \( a e^{-2t} \)

\[
\frac{d^2(e^{-2t})}{dt^2} + 5 \frac{d(e^{-2t})}{dt} + 6 (ae^{-2t})
\]

\[
= (4 e^{-2t} - 10 e^{-2t}) + 6 (e^{-2t})
\]

\[ = 0 \]

We guessed a solution to the homogeneous eqn.

Stuck? No, use \( y_p(t) = te^{-2t} \)

Works \( \checkmark \) (try)

Step 3

\[ te^{-2t} + C_1 e^{-2t} + C_2 e^{-3t} \]

is general solution.

(ex 3) Sinusoidal forcing

\[
\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 2y = \sin(t)
\]
Step 1  Assoc’d homog. eqn.
\[ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 2y = 0 \]
char. eqn.  \[ r^2 + 2r + 2 = 0 \]
roots  \[ -1 \pm i \]  underdamped

\[ C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t) \]

Step 2  Could guess particular soln.
\[ A \cos(t) + B \sin(t) \], solve
for  \[ A \neq B \], but there’s
a simpler way

\[ \sin(t) \]  is  \[ \text{Im}(e^{it}) \]  by Euler’s formula.

Solve \[ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 2y = e^{it} \]
for complex-valued  \[ y \]  “complexification”
Then \[ \frac{d^2}{dt^2} \left( y_{\text{Re}} + i y_{\text{Im}} \right) + 2 \frac{d}{dt} \left( y_{\text{Re}} + i y_{\text{Im}} \right) + 2 \left( y_{\text{Re}} + i y_{\text{Im}} \right) \]

taking real \[ \cos(t) \]
imag \[ \sin(t) \]

\[ \frac{d^2 y_{\text{Re}}}{dt^2} + 2 \frac{d y_{\text{Re}}}{dt} + 2 y_{\text{Re}} = \cos(t) \]

\[ \frac{d^2 y_{\text{Im}}}{dt^2} + 2 \frac{d y_{\text{Im}}}{dt} + 2 y_{\text{Im}} = \sin(t) \]

Hence \( y_{\text{Im}}(t) \) is a particular solution to original prob w/ forcing \( \sin(t) \)

Why is it easier?

Guess a \( e^{it} \) for complex number \( a \)

\[ \frac{d^2 (ae^{it})}{dt^2} + 2 \frac{d (ae^{it})}{dt} + 2 (ae^{it}) \]

\[ = ae^{-it} \left( -e^{it} + i2e^{it} + 2e^{it} \right) \]
\[ y(t) = a \left(1 + \frac{2i}{5}\right)e^{ct} \]

Worked for \( a = \frac{1}{1 + 2i} = \frac{1 - 2i}{5} \)

Now,

\[ y(t) = \frac{1 - 2i}{5} \left(\cos(t) + i \sin(t)\right) \]

\[ = \frac{1}{5} \cos(t) - \frac{2}{5} \sin(t) + i\left(\frac{1}{5} \sin(t) - \frac{2}{5} \cos(t)\right) \]

im\mbox{aginary part:}

\[ y_{\text{ip}}(t) = \frac{1}{5} \sin(t) - \frac{2}{5} \cos(t) \]

Step 3

\[ y(t) = \frac{1}{5} \sin(t) - \frac{2}{5} \cos(t) + c_1 e^{-t} \sin(t) + c_2 e^{-t} \cos(t) \]
again, solve approach $y_p(t)$

Unforced response tends to 0

with positive $m, b, k$

\[ C_1 y_1(t) + C_2 y_2(t) + y_p(t) \Rightarrow y_p(t) \]

unforced soln forced soln

as $t \to \infty$

At long times, there is no

memory of the initial condi-

tions $y_p(t)$ is called the forced response,

or the steady-state response

In-class exercise

\[ \text{solve } d^2 y \over dt^2 + dy \over dt + y = 0 \]
\[
\frac{d^2t}{dt^2} + \frac{dt}{dt} + 10 y = 1 \cos \omega t
\]